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Integral Operator and Starlike Functions

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Abstract—We define a class of univalent starlike functions and consider the following integral operator:

$$F(z) = \frac{\gamma+1}{z^\gamma} \int_0^z f(t)t^{\gamma-1}\,dt, \qquad \gamma \geq 0. \tag{$*$}$$

It is well known that, if f is starlike, then F is also starlike. We extend this result for a more general class.

Keywords—Starlike functions, α-convex functions, Integral operator, Holomorphic function.

1. INTRODUCTION

Let U be the unit disc of the complex plane

$$U = \{z \in \mathbb{C},\ |z| < 1\},$$

and let $\mathcal{H}[U]$ be the space of holomorphic functions in U. We let

$$A = \left\{f \in \mathcal{H}[U],\ f(z) = z + a_2z^2 + a_3z^3 + \cdots,\ z \in U\right\},$$
$$A_n = \left\{f \in \mathcal{H}[U],\ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots,\ z \in U\right\},$$

where $A_1 = A$, and

$$S^* = \left\{f \in A,\ \operatorname{Re}\frac{zf'(z)}{f(z)} > 0,\ z \in U\right\}$$

denote the class of starlike functions in U.

We recall that, see [1],

$$M_\alpha = \left\{f \in A,\ \operatorname{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f(z)} + 1\right)\right] > 0,\ z \in U\right\}$$

is the class of α-convex functions. Further (see [2]),

$$S^*(\alpha) = \left\{f \in A,\ \operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha,\ z \in U\right\}$$

is the class of starlike functions of order α, $\alpha < 1$.

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If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$ for all $z \in U$ such that $f(z) = g(w(z))$ for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

The following statements can be found in [1].

STATEMENT 1. If $\alpha \in \mathbb{R}$ and $f \in M_\alpha$, then f is a starlike function.

STATEMENT 2. If $\alpha \geq 0$ and $f \in A$, then $f \in M_\alpha$ if and only if

$$F(z) = f(z) \left[\frac{zf'(z)}{f(z)}\right]^\alpha$$

is starlike.

Let $\alpha \in \mathbb{C}$ and $f \in A$. We say that the function f is α-starlike if the function $F : U \to \mathbb{C}$, where

$$F(z) = f(z) \left(1 - \alpha + \alpha \frac{zf'(z)}{f(z)}\right), \qquad z \in U,$$

is starlike [3].

In [4], the following result is proven.

LEMMA 1. *Let $\psi : \mathbb{C}^2 \to \mathbb{C}$, and ψ satisfies the condition*

$$\operatorname{Re} \psi(is, t) \leq 0,$$

for $s \in \mathbb{R}$, $t \leq -(1/2)(1 + s^2)$. Moreover, if $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ satisfies

$$\operatorname{Re} \psi[p(z), zp'(z)] > 0,$$

then

$$\operatorname{Re} p(z) > 0.$$

A more general form of this lemma can be found in [4].

In [5,6], the authors proved the next result.

LEMMA 2. *Let q be univalent in U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$, when $w \in q(U)$. Set*

$$Q(z) = zq'(z)\phi[q(z)],$$
$$h(z) = \theta[q(z)] + Q(z),$$

and suppose that

(i) *Q is starlike, and*

(ii) $\operatorname{Re} zh'(z)/Q(z) = \operatorname{Re} [\theta'[q(z)]/\phi[q(z)] + zQ'(z)/Q(z)] > 0$.

If p is analytic in U, with $p(0) = q(0)$, $p(U) \subset D$, and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)] = h(z),$$

then $p \prec q$, and q is the best dominant.

For real α and β in [7], I introduced the class $M_{\alpha,\beta}$ for functions $f \in A$, with the properties

$$\frac{f(z)f'(z)}{z} \neq 0, \qquad 1 - \alpha + \alpha \frac{zf'(z)}{f(z)} \neq 0,$$

and assuming that the function

$$F(z) = f(z) \left[\frac{zf'(z)}{f(z)}\right]^{\alpha(1-\beta)} \left[1 - \alpha + \alpha \frac{zf'(z)}{f(z)}\right]^\beta \tag{1}$$

is starlike. I proved that f is also starlike for $0 < \alpha \leq 1$ and $\beta \geq 0$. In this paper, I improve this result by showing that f is starlike if F is starlike of some negative order. Further, I prove that $F \in S^*$ where F is given by $(*)$.

2. MAIN RESULTS

DEFINITION. *Let $0 \le \alpha \le 1$, $\beta \ge 0$, and let $f \in A$, with*

$$\frac{f(z)f'(z)}{z} \neq 0, \quad 1-\alpha+\alpha\frac{zf'(z)}{f(z)} \neq 0, \qquad z \in U;$$

then we say that the function f belongs to the class $\tilde{M}_{\alpha,\beta}$ if the function $F : U \to \mathbb{C}$ defined by (1) is starlike of order $C(\alpha,\beta)$, where

$$C(\alpha,\beta) = \begin{cases} -\dfrac{\alpha\beta}{2(1-\alpha)}, & \text{if } 0 \le \alpha \le \dfrac{1}{2}, \\ -\dfrac{1-\alpha}{2\alpha}\beta, & \text{if } \dfrac{1}{2} \le \alpha \le 1. \end{cases} \tag{2}$$

THEOREM 1. *For any $\alpha \in [0,1]$, $\beta \ge 0$, the inclusion $\tilde{M}_{\alpha,\beta} \subset S^*$ holds.*

PROOF. Let $p(z) = zf'(z)/f(z)$, $z \in U$; then condition $f(z)f'(z)/z \neq 0$ gives that the function p is holomorphic in U. Since F satisfies

$$\operatorname{Re}\frac{zF'(z)}{F(z)} > C(\alpha,\beta), \tag{3}$$

with $C(\alpha,\beta)$ given by (2), from (1), we deduce

$$\begin{aligned}\frac{zF'(z)}{F(z)} - C(\alpha,\beta) &= p(z) + \alpha(1-\beta)\frac{zp'(z)}{p(z)} + \alpha\beta\frac{zp'(z)}{1-\alpha+\alpha p(z)} - C(\alpha,\beta) \\ &= \psi[p(z), zp'(z)].\end{aligned}$$

Condition (3) is equivalent to

$$\operatorname{Re}\psi[p(z), zp'(z)] > 0.$$

We have

$$\begin{aligned}\operatorname{Re}\psi(is,t) &= \operatorname{Re}\left[is + \alpha(1-\beta)\frac{t}{is} + \alpha\beta\frac{t}{1-\alpha+\alpha is}\right] - C(\alpha,\beta) \\ &= \frac{\alpha\beta t(1-\alpha)}{(1-\alpha)^2+\alpha^2 s^2} - C(\alpha,\beta) \\ &\le -\frac{\alpha\beta(1-\alpha)\left(1+s^2\right)}{2\left[(1-\alpha)^2+\alpha^2 s^2\right]} - C(\alpha,\beta) \\ &\le -\left[\frac{\alpha\beta(1-\alpha)\left(1+s^2\right)}{2\left[(1-\alpha)^2+\alpha^2 s^2\right]} + C(\alpha,\beta)\right] \le 0.\end{aligned}$$

Hence, by Lemma 1, we deduce $\operatorname{Re} p(z) > 0$, which shows that $f \in S^*$.

THEOREM 2. *Let $\gamma \ge 0$, and let*

$$h(z) = \frac{1+z}{1-z} + \frac{2z}{(1-z)[1+\gamma+(1-\gamma)z]}, \qquad z \in U. \tag{4}$$

If $f \in A$ and

$$\frac{zf'(z)}{f(z)} \prec h(z), \tag{5}$$

then $F \in S^*$, where F is given by $(*)$.

PROOF. From (*), one can get

$$\gamma F(z) + zF'(z) = (\gamma + 1) f(z). \tag{6}$$

If we let $p(z) = zF'(z)/F(z)$, then (6) becomes

$$\begin{aligned} \gamma F(z) + p(z)F(z) &= (\gamma + 1) f(z), \\ [\gamma + p(z)]F(z) &= (\gamma + 1) f(z), \end{aligned}$$

and we have

$$\frac{zp'(z)}{p(z)+\gamma} + p(z) = \frac{zf'(z)}{f(z)}.$$

Then, (5) becomes

$$p(z) + \frac{zp'(z)}{p(z)+\gamma} \prec h(z).$$

To show that $F \in S^*$, we shall use Lemma 2. Let us introduce the functions

$$q(z) = \frac{1+z}{1-z}, \qquad \theta(w) = w, \qquad \phi(w) = \frac{1}{w+\gamma};$$

then by simple algebra we get

$$\begin{aligned} \theta[q(z)] &= q(z) = \frac{1+z}{1-z}, \\ \phi[q(z)] &= \frac{1}{q(z)+\gamma} = \frac{1-z}{1+\gamma+(1-\gamma)z}, \\ Q(z) &= zq'(z)\phi[q(z)] = \frac{2z}{(1-z)[1+\gamma+(1-\gamma)z]}, \\ h(z) &= \theta[q(z)] + Q(z) = \frac{1+z}{1-z} + \frac{2z}{(1-z)[1+\gamma+(1-\gamma)z]}. \end{aligned}$$

Since Q is starlike and $\operatorname{Re}\phi(q(z)) > 0$, by Lemma 2 we deduce $p \prec q$, i.e., $F \in S^*$.

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